SOLUTION OF TWO-DIMENSIONAL PROBLEMS OF FILTRATION WITH A LIMITING GRADIENT BY THE METHOD OF SMALL PARAMETER

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One of the authors has shown in [1] that, for a number of symmetrical configurations of sources and sinks, the stream function ψ regarded as a function of the point (w, θ) in the plane of the filtration rate hodograph (w is the rate of filtration and θ is the angle made made by it with the x-axis), satisfies Eq.

$$w(w+\lambda)\frac{\partial^2\psi}{\partial w^2}+(w-\lambda)\frac{\partial\psi}{\partial w}+\frac{\partial^2\psi}{\partial\theta^2}=0 \qquad (0.1)$$

in a semi-infinite strip $0<w<\infty$, $0<\theta<\theta_0$ with the exception of a segment $0<w<a$; $\theta=\theta_1$ or a ray $a<w<\infty$, $\theta=\theta_1$.

Here λ denotes the characteristic rate proportional to the value of the limiting gradient, the latter given by the condition that, if the modulus of the pressure gradient falls below the limiting value in some region, then there will be no motion within that region (stagnation zone). The problem thus reduces to the determination of the velocity field around the sources and sinks and of the boundaries of resulting stagnation zones. No exact solution of this problem has been obtained for an arbitrary value of a.

When the filtration law is linear ($\lambda=0$), Eq. (0.1) becomes a Laplace's equation in polar coordinates (w, θ), and its solution is easily obtained by means of a conformal transformation. This suggests that, when λ is small, then expansion in a small parameter may offer a possible method of solution of (0.1). However, the method of small parameter cannot be applied directly to (0.1), since λ appears in the coefficient of the higher derivative and becomes zero when $w=0$, i.e. on the boundaries of stagnation zones.

Below we give a method of constructing an expansion in the small parameter λ, which is valid throughout. To simplify the calculations, we shall consider the case of two sources of equal intensity (see e.g. [1]), in which additional symmetry reduces the mapping on the hodograph plane to a semistrip of width $\theta_1=\frac{1}{2}\pi$. The method is, nevertheless, applicable to the general case.

1. Formulation of the problem and its reduction to a singular integral equation. 1°. We shall consider a problem corresponding to a flow emerging from two sources of equal intensity, separated by the distance of $2L$.

We shall use the Cartesian xy-coordinate system with the origin at the center of symmetry and the x-axis passing through the sources. General considerations [1] infer that a symmetrical stagnation zone appears in this problem. This zone is bounded by four arcs and the resulting figure resembles an astroid with its vertices, which are cusp points, lying on the coordinate axes. By symmetry we need only consider the flow in the second quadrant. The corresponding problem on the hodograph plane consists of the determination of the stream function ψ as a solution of (0.1) in the semistrip $0<\theta<\frac{1}{2}\pi$, $0<w<\infty$, under the following conditions:

$$\psi(w,0)=0,\quad \psi(0,\theta)=0$$

$$\psi\left(w,\frac{\pi}{2}\right)=0\quad (w<a),\qquad \frac{\partial\psi(w,\frac{1}{2}\pi)}{\partial\theta}=\frac{Q}{\pi}\quad (a<w<\infty) \qquad (1.1)$$

The velocity w becomes zero at the tip of the stagnation zone lying on the y-axis, and when $y \to \infty$, therefore it passes through a maximum. The magnitude a (previously undefined) which appears in the formulation of (1.1), represents the maximum value of the filtration rate on the axis of symmetry in the physical plane, perpendicular to the line connecting the sources. We find it more convenient to fix the value of a and find the corresponding value of L.

Let us introduce the dimensionless filtration rate

$$u = w / \lambda \tag{1.2}$$

as the independent variable. The (0.1) becomes

$$u(u+1)\frac{\partial^2\psi}{\partial u^2} + (u-1)\frac{\partial\psi}{\partial u} + \frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{1.3}$$

with the boundary conditions

$$\psi(u,0) = \psi(0,\theta) = 0, \quad \psi(u, {}^1/_2\pi) = 0 \quad (u \leqslant a_0 = a/\lambda)$$

$$\frac{\partial\psi(u, {}^1/_2\pi)}{\partial\theta} = \frac{Q}{\pi\lambda} \qquad (a_0 < u < \infty) \tag{1.4}$$

Let us now apply to (1.3) the corresponding integral transformation discussed in [2]. For the transform $\psi^*(\alpha, \theta)$ we have

$$\psi^*(\alpha,\theta) = \int_0^\infty (1+u) F(2+i\sqrt{\alpha},\ 2-i\sqrt{\alpha},\ 3,\ -u)\,\psi(u,\theta)\,du \tag{1.5}$$

where F is a hypergeometric function. In the following we shall denote, for brevity,

$$F(2+i\sqrt{\alpha}, 2-i\sqrt{\alpha},\ 3,\ -u) \equiv F(\alpha, -u) \tag{1.6}$$

Multiplying (1.3) by $(1+u)F(\alpha,-u)$ and integrating by parts we obtain, taking into account the conditions at $w=0$ and $w=\infty$

$$\frac{d^2\psi^*}{d\theta^2} - \alpha\psi^* = 0 \tag{1.7}$$

Solution of (1.7) which becomes zero when $\theta = 0$, has the form

$$\psi^*(\alpha,\theta) = A(\alpha)\ \mathrm{sh}\ \sqrt{\alpha}\theta \tag{1.8}$$

and the transformation formula (1.7) of [2] gives

$$\psi(u,\theta) = \frac{1}{4}u^2\int_0^\infty \alpha(1+\alpha)\frac{\mathrm{sh}\sqrt{\alpha}\,\theta}{\mathrm{th}\sqrt{\alpha}\,\pi}A(\alpha)F(\alpha,-u)\,d\alpha \tag{1.9}$$

2°. Let us put

$$\psi(u, {}^1/_2\pi) = g(u), \quad a_0 < u < \infty \tag{1.10}$$

Then we obviously have

$$A(\alpha)\,\mathrm{sh}\,{}^1/_2\pi\sqrt{\alpha} = \int_{a_0}^\infty (1+u)\,g(u)\,F(\alpha,-u)\,du \tag{1.11}$$

Inserting (1.11) into (1.9) and substituting the resulting expression into the last boundary condition of (1.4), we obtain the following equation for $g(u)$

$$u^2\int_0^\infty \frac{\alpha^{3/2}(1+\alpha)}{\mathrm{th}\sqrt{\alpha}\,\pi\,\mathrm{th}\,{}^1/_2\sqrt{\alpha}\,\pi}F(\alpha,-u)\,d\alpha\int_{a_0}^\infty(1+v)\,g(v)\,\times$$

$$\times F(\alpha,-v)\,dv = -\frac{4Q}{\pi} \equiv \int_{a_0}^\infty K(u,v)\,g(v)\,dv \tag{1.12}$$

Putting $\alpha = s^2$ we obtain the following expression for the kernel $K(u, v)$

$$K(u, v) = 2u^2(1+v)\int_0^\infty \frac{s^4(1+s^2)}{\operatorname{th} s\pi \operatorname{th} {}^1/_2 s\pi} F(s^2, -u) F(s^2, -v)\, ds \qquad (1.13)$$

which can be transformed by means of the following formula (see e.g. [3]):

$$F(2+is,\ 2-is,\ 3,\ -u) = \frac{\Gamma(3)\Gamma(-2is)}{\Gamma(2-is)\Gamma(1-is)} u^{-2-is} \times \qquad (1.14)$$

$$\times F\left(2+is,\quad is,\ 1+2is, -\frac{1}{u}\right) + \frac{\Gamma(3)\Gamma(2is)}{\Gamma(2+is)\Gamma(1+is)} u^{-2+is} \times$$

$$\times\ F(2-is,\ -is,\ 1-2is,\ -1/u)$$

Inserting (1.14) into (1.13) and assuming that the parity of the factor $F(s^2,\ -u)$ in the integrand is a function of s, we obtain

$$K(u, v) = 4u^2(1+v)\int_{-\infty}^{\infty} \frac{s^4(1+s^2)}{\operatorname{th} s\pi \operatorname{th}{}^1/_2 s\pi} \times$$

$$\times \frac{\Gamma(-2is)}{\Gamma(2-is)\Gamma(1-is)} \frac{F_+(-1/u)}{u^{2+is}} F(s^2,\ -v)\, ds \qquad (1.15)$$

where

$$F_\pm(-1/u) = F(2 \pm is, \pm is,\ 1 \pm 2is, -1/u) \qquad (1.16)$$

Let us now apply Formula (1.14) to the function $F(s^2,\ -v)$, making at the same time the substitution $\xi = is$ in (1.15). We obtain

$$K(u, v) = K_1(u, v) + K_2(u, v) \qquad (1.17)$$

where

$$K_1(u, v) = \frac{2(1+v)}{\pi i v^2} \int_{-i\infty}^{i\infty} \xi \operatorname{ctg} \frac{\pi\xi}{2} \left(\frac{v}{u}\right)^\xi F_+\left(-\frac{1}{u}\right) F_-\left(-\frac{1}{v}\right) d\xi$$

$$K_2(u, v) = \frac{2(1+v)}{\pi i v^2} \times \qquad (1.18)$$

$$\times \int_{-i\infty}^{i\infty} \xi \operatorname{ctg} \frac{\pi\xi}{2} \frac{\Gamma(-2\xi)\Gamma(2+\xi)\Gamma(1+\xi)}{\Gamma(2\xi)\Gamma(2-\xi)\Gamma(1-\xi)} \frac{F_+(1/u)F_+(-1/v)}{(uv)^\xi} d\xi$$

3°. In the following we shall utilize the asymptotic expressions for $F_\pm$. By the Euler's formula [3] we have

$$u^{-z}F\left(2+z,\ z,\ 1+2z, -\frac{1}{u}\right) = \frac{\Gamma(1+2z)u^2}{\Gamma(z)\Gamma(1+z)} \int_0^1 \left(\frac{t(1-t)}{u+t}\right)^z \frac{dt}{(u+t)^2 t} \qquad (1.19)$$

Let us introduce a new variable

$$\nu = \frac{t(1-t)}{u+t}, \qquad t_\pm = \frac{1-\nu}{2} \pm \frac{1}{2}\sqrt{(1-\nu)^2 - 4u\nu} \qquad (1.20)$$

Then the integral in the right-hand side of (1.19) becomes

$$I = \frac{1}{u(1+u)} \int_0^n \frac{\nu^{z-1}(1+\nu)\, d\nu}{\sqrt{(1-\nu)^2 - 4\nu u}} + \frac{1}{u^2(1+u)} \int_0^n \nu^{z-1} \sqrt{(1-\nu)^2 - 4u\nu}\, d\nu \qquad (1.21)$$

where $n(u)$ denotes the value of $\nu(t)$, which reaches its maximum on the segment $(0, 1)$ and is given by $n(u) = 1 + 2u - 2\sqrt{u(u+1)}, \qquad u = (1-n)^2/4n \qquad (1.22)$

Let us now put $\nu = \eta n$ in the integrals of (1.21) and expand the expressions under the radical signs in powers of $n^2(1-\eta)/(1-n^2)$ which is less than unity for $0 \leqslant \eta \leqslant 1$

and all $u > a_0$ provided that $a_0 > 1/_8$ (λ is small). Then the termwise integration yields

$$I = \frac{16n^{z+2}}{(1-n^2)^2}\left(\frac{1+n}{1-n}\right)^{1/2}\frac{\Gamma(1/2)\,\Gamma(z)}{\Gamma(z+1/2)}\left[1+\frac{3}{2}\frac{n}{z+1/2}+O(n^3)\right]$$

consequently

$$u^{-z}F(2+z,\ z,\ 1+2z,\ -1/u) = (4n)^z\left(\frac{1-n}{1+n}\right)^{3/2}\left(1+\frac{3}{2}\frac{n}{z+1/2}+O(n^2)\right] \qquad (1.23)$$

The above expression makes it possible to isolate the principal singular part of the kernel K. Putting

$$K_0(u,\ v) \equiv K_0(n,\ m) = \frac{2(1+v)}{\pi i v^2}\left(\frac{1-n}{1+n}\right)^{3/2}\left(\frac{1-m}{1+m}\right)^{3/2}\int_{-i\infty}^{i\infty}\xi\operatorname{ctg}\frac{\pi\xi}{2}\left(\frac{n}{m}\right)^{\xi}d\xi$$

$$m(v) = 2v+1-2\sqrt{v(1+v)}, \qquad v = (1-m)^2/4m \qquad (1.24)$$

let us represent the integral equation (1.12) in the form

$$\int_{a_0}^{\infty} g(v)\,K_0(u,\ v)\,dv = -\frac{4Q}{\pi} - \int_{a_0}^{\infty} g(v)\,[K(u,\ v) - K_0(u,\ v)]\,dv \equiv f(u) \quad (1.25)$$

or, in the n, m-variables,

$$\frac{2}{\pi i}\int_0^M\left(\frac{1+m}{1-m}\right)^{3/2}G(m)\int_{-i\infty}^{i\infty}\xi\operatorname{ctg}\frac{\pi\xi}{2}\left(\frac{n}{m}\right)^{\xi}d\xi\,\frac{dm}{m} =$$

$$= F(n) \equiv \left(\frac{1+n}{1-n}\right)^{3/2} f\left(\frac{(1-n)^2}{4n}\right) \qquad (1.26)$$

where
$$G(m) = g\left(\frac{(1-m)^2}{4m}\right), \quad M = m(a_0) = 1+2a_0-2\sqrt{a_0(a_0+1)} \quad (1.27)$$

By (1.27) we have $M \sim 1/(4a_0) \to 0$ as $\lambda \to 0\ (a_0 \to \infty)$.

2. Solution of the auxiliary problem. We shall consider an auxiliary problem obtained by putting the limit gradient in the initial problem equal to zero ($\lambda = 0$). Let Eq.

$$u^2\frac{\partial^2\chi}{\partial u^2} + u\frac{\partial\chi}{\partial u} + \frac{\partial^2\chi}{\partial\theta^2} = 0 \qquad (2.1)$$

be given on the semistrip $0 < \theta < 1/_2\pi$, $0 < w < \infty$, with the conditions

$$\chi(u,\ 0) = 0;\ \chi(0,\ \theta) = 0,$$

$$\chi(u,\ 1/_2\pi) = 0 \quad (0 < u < U) \qquad \partial\chi\ (u,\ 1/_2\pi)\,/\,\partial\theta = \varphi(u) \qquad (U < u < \infty) \qquad (2.2)$$

and let $\varphi(u)$ satisfy

$$\lim_{u\to\infty}\varphi(u) = Q/\pi \qquad (2.3)$$

Putting

$$\chi(u,\ 1/_2\pi) = P(u) \qquad (u > U) \qquad (2.4)$$

applying to (2.1) the Mellin transform [4] and repeating the procedure of Section 1, we easily obtain, for the function $P(u)$, the following integral equation:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}u^{-\xi}\,\xi\operatorname{ctg}\frac{\pi\xi}{2}\,d\xi\int_U^{\infty}v^{\xi-1}P(v)\,dv = \varphi(u) \qquad (2.5)$$

Putting now $u = 1/n$ and $v = 1/m$, we obtain

$$\frac{1}{2\pi i}\int_0^{M'}\frac{P^*(m)}{m}\,dm\int_{-i\infty}^{i\infty}\left(\frac{n}{m}\right)^{\xi}\xi\operatorname{ctg}\frac{\pi\xi}{2}\,d\xi = \varphi^*(n) \qquad (2.6)$$

$$(P^*(n) = P(1/n),\ \varphi^*(n) = \varphi(1/n),\ M' = 1/U)$$

which coincides, to within the accuracy of notation used, with (1.26) (with one exception, namely, that the first part of (1.26) is dependent on an unknown function).

Solution of the problem (2.1) – (2.3) is easily obtained by considering (2.1) as a Laplace's equation in polar coordinates. Putting $z = we^{i\theta}$ we obtain, for χ, a mixed boundary value problem in the first quadrant of the z-plane. On mapping it onto a semiplane, we obtain the required sopution using the Keldysh-Sedov formula [5]. As the result, we have

$$P(u) = \frac{Q}{\pi} \operatorname{arc\,tg} \sqrt{\frac{u^2}{U^2} - 1} - \frac{1}{2\pi} \int_0^\infty \frac{\varphi(\sqrt{s + U^2}) - Q/\pi}{s + U^2} \ln \left| \frac{\sqrt{s} - \sqrt{u^2 - U^2}}{\sqrt{s} + \sqrt{u^2 - U^2}} \right| ds \tag{2.7}$$

3. Reduction to the integral Fredholm equation. 1°. Comparing (2.6) with (1.26) we find that when $M \to M'$, $[(1+m)/(1-m)]^{3/2} G(m) \to P^*(m)$, and $F(n) \to 4\varphi^*(n)$ the above equations become identical. Use of Formula (2.17) to solve (1.26) yields

$$\left(\frac{1+m}{1-m}\right)^{3/2} G(m) = \frac{Q}{\pi} \operatorname{arc\,tg} \left(\frac{M^2}{m^2} - 1\right)^{1/2} - \tag{3.1}$$

$$- \frac{M^2}{2\pi} \int_0^\infty \left[\frac{1}{4} F\left(\frac{M}{\sqrt{1 + sM^2}}\right) - \frac{Q}{\pi}\right] \ln \left| \frac{mM\sqrt{s} - \sqrt{M^2 - m^2}}{mM\sqrt{s} + \sqrt{M^2 - m^2}} \right| \frac{ds}{1 + sM^2}$$

Substitution of the expression for F, in accordance with (1.26) and (1.27), gives

$$\left(\frac{1+m}{1-m}\right)^{3/2} G(m) = \frac{Q}{\pi} \operatorname{arctg} \left(\frac{M^2}{m^2} - 1\right)^{1/2} - \frac{QM^2}{2\pi^2} \int_0^\infty \left[\left(\frac{\sqrt{sM^2+1} + M}{\sqrt{sM^2+1} - M}\right)^{3/2} - 1\right] \times$$

$$\times \ln \left| \frac{mM\sqrt{s} - \sqrt{M^2 - m^2}}{mM\sqrt{s} + \sqrt{M^2 - m^2}} \right| \frac{ds}{1 + sM^2} + \frac{M^2}{32\pi} \int_0^M G(\sigma) \frac{1 - \sigma^2}{\sigma^2} d\sigma \times$$

$$\times \left\{ \int_0^\infty \left[K^*\left(\frac{M}{\sqrt{1 + sM^2}}, \sigma\right) - K_0^*\left(\frac{M}{\sqrt{1 + sM^2}}, \sigma\right)\right] \ln \left| \frac{mM\sqrt{s} - \sqrt{M^2 - m^2}}{mM\sqrt{s} + \sqrt{M^2 - m^2}} \right| \times \right.$$

$$\left. \times \left(\frac{\sqrt{sM^2+1} + M}{\sqrt{sM^2+1} - M}\right)^{3/2} \frac{ds}{1 + sM^2} \right\} \tag{3.2}$$

Below we shall consider the integral equation (3.2) only for small values of M (which corresponds to $\lambda \to 0$), making at the same time the independent variable proportional to M. Introducing new variables

$$\zeta = m/M, \quad Y(\zeta) = [(1 + M\zeta)/(1 - M\zeta)]^{3/2}\ G(M\zeta) \tag{3.3}$$

and changing the variables in the inner integration, we obtain

$$Y(\zeta) = \frac{Q}{\pi} \operatorname{arc\,tg} \frac{\sqrt{1 - \zeta^2}}{\zeta} - \frac{Q}{2\pi^2} \int_0^\infty \left[\left(\frac{\sqrt{s+1} + M}{\sqrt{s+1} - M}\right)^{3/2} - 1\right] \times$$

$$\times \ln \left| \frac{\zeta\sqrt{s} - \sqrt{1 - \zeta^2}}{\zeta\sqrt{s} + \sqrt{1 - \zeta^2}} \right| \frac{ds}{1 + s} + \frac{1}{32\pi M} \int_0^1 Y(\sigma) \frac{(1 - M\sigma)^{5/2}}{\sigma^2 (1 + M\sigma)^{1/2}} d\sigma \times$$

$$\times \left\{ \int_0^\infty \left[K^*\left(\frac{M}{\sqrt{s+1}}, M\sigma\right) - K_0^*\left(\frac{M}{\sqrt{s+1}}, M\sigma\right)\right] \ln \left| \frac{\zeta\sqrt{s} - \sqrt{1 - \zeta^2}}{\zeta\sqrt{s} + \sqrt{1 - \zeta^2}} \right| \times \right.$$

$$\times\left(\frac{\sqrt{s+1}+M}{\sqrt{s+1}-M}\right)^{1/2}\frac{ds}{1+s}\Big\} \equiv A(\zeta)+\int_0^1 Y(\sigma)R(\zeta,\sigma)\,d\sigma \qquad (3.4)$$

2°. Let us consider the difference $K^* - K_0^*$ in more detail. By (1.17),(1.18) and (1.24) we have

$$K^*\left(\frac{M}{\sqrt{s+1}},\ M\sigma\right)-K_0^*\left(\frac{M}{\sqrt{s+1}},\ M\sigma\right)=$$

$$=\frac{2(1+v)}{\pi i v^2}\int_{-i\infty}^{i\infty}\xi\,\mathrm{ctg}\,\frac{\pi\xi}{2}\left[\left\{\left(\frac{v}{u}\right)^{\xi}F_+\left(-\frac{1}{u}\right)F_-\left(-\frac{1}{v}\right)-\left(\frac{n}{m}\right)^{\xi}\left(\frac{(1-n)(1-m)}{(1+n)(1+m)}\right)^{1/2}\right]+\right.$$

$$\left.+(uv)^{-\xi}F_+\left(-\frac{1}{u}\right)F_+\left(-\frac{1}{v}\right)\right\}d\xi \qquad (3.5)$$

$$u=\frac{1}{4}\left(1-\frac{M}{\sqrt{s+1}}\right)^2\frac{\sqrt{s+1}}{M},\qquad v=\frac{1}{4}\,\frac{(1-M\sigma)^2}{M\sigma}$$

which we shall now consider at small M. Utilizing the fact that the expansions (1.23) hold (uniformly in ξ) for the hypergeometric functions, we obtain

$$K^*\left(\frac{M}{\sqrt{s+1}},\ M\sigma\right)-K_0^*\left(\frac{M}{\sqrt{s+1}},\ M\sigma\right)=\frac{8M\sigma}{\pi i}\left\{\frac{3}{2}M\int_{-i\infty}^{i\infty}\xi\,\mathrm{ctg}\,\frac{\pi\xi}{2}\,(\sigma\sqrt{s+1})^{-\xi}\times\right.$$

$$\times\left(\frac{1}{\sqrt{s+1}\,(\xi+1/2)}+\frac{\sigma}{1/2-\xi}+O(M)\right)d\xi\,+$$

$$\left.+\int_{-i\infty}^{i\infty}\xi\,\mathrm{ctg}\,\frac{\pi\xi}{2}\left[\frac{16M^2}{\sigma\sqrt{s+1}}\right]^{\xi}(1+O(M))\,\frac{\Gamma(-2\xi)\,\Gamma(2+\xi)\,\Gamma(1+\xi)}{\Gamma(2\xi)\,\Gamma(2-\xi)\,\Gamma(1-\xi)}\,d\xi\right\} \qquad (3.6)$$

Second integral in (3.6) can be converted into a line integral along a staight line parallel to the imaginary axis and situated to the right of it at the distance c, $1/2<c<3/2$, by adding the contribution from the pole of the integrand at $\xi = 1/2$. The following estimate is valid for the line integral:

$$\left|\int_{c-i\infty}^{c+i\infty}\right| \leqslant \mathrm{const}\, M^{2c}$$

Thus the intergral along the imaginary axis is equal, to within the required accuracy, to the residue at the pole

$$\int_{-i\infty}^{i\infty} = -\frac{3}{2}\pi i\sigma^{1/2}(s+1)^{-1/4}M+O(M^2) \qquad (3.7)$$

The first integral of (3.6) can also, depending on the sign of $\ln(\sigma\sqrt{s+1})$, be represented as a sum of residues of the integrand in the right or left semiplane. As the result we obtain

$$\frac{3}{2}M\int_{-i\infty}^{i\infty} = \frac{3}{2}\,\frac{M\pi i\sqrt{\sigma}}{(s+1)^{1/4}}-\sum_{n=1}^{\infty}\frac{12M\pi i}{(\sigma\sqrt{s+1})^{2n}}\left[\frac{n}{(4n+1)\sqrt{s+1}}+\frac{\sigma n}{1-4n}\right]$$

$$(\sigma\sqrt{s+1}>1) \qquad (3.8)$$

$$\frac{3}{2}M\int_{-i\infty}^{i\infty} = \frac{3}{2}\,\frac{M\pi i\sqrt{\sigma}}{(s+1)^{1/4}}-\sum_{n=1}^{\infty}12M\pi i\,(\sigma\sqrt{s+1})^{2n}\left[\frac{n}{(1-4n)\sqrt{s+1}}+\frac{\sigma n}{1+4n}\right]$$

$$(\sigma\sqrt{s+1}<1) \qquad (3.9)$$

Taking (3.7) into account we find, that the difference $K^* - K_0^*$ is, within the approximation considered, equal to the series appearing in the right-hand sides of (3.8) or (3.9) depending on the sign of $\sigma \sqrt{s+1}$. Thus we have

$$K^*(M(s+1)^{-1/2}, M\sigma) - K_0^*(M(s+1)^{-1/2}, M\sigma) =$$
$$= -\frac{16M\sigma}{\pi^2 i}\left(\frac{12M\pi i}{\sqrt{s+1}} f(\sigma\sqrt{s+1}) + O(M^2)\right) \qquad (3.10)$$

where

$$f(z) = \sum_{n=1}^{\infty} n z^{2n}\left(\frac{z}{4n+1} - \frac{1}{4n-1}\right) \qquad (z < 1)$$

$$f(z) = \sum_{n=1}^{\infty} n z^{-2n}\left(\frac{1}{4n+1} - \frac{z}{4n-1}\right) \qquad (z > 1)$$

so that

$$f(z) = \frac{1}{4}\left[\frac{z}{1+z} - \frac{\sqrt{z}}{2}\ln\left|\frac{1+\sqrt{z}}{1-\sqrt{z}}\right|\right] \qquad (3.11)$$

which becomes, on substitution $z = s^2$, an elementary sum of a geometrical progression.

3°. Solution of the integral equation (3.4) at small M can be obtained by the method of consecutive approximations. Using (3.11) we find, that a solution is given, with the accuracy of up to the terms of the order of M^2, by

$$Y(\zeta) = \frac{Q}{\pi}\arccos\zeta - \frac{3QM}{2\pi^2}\int_0^\infty \ln\left|\frac{\zeta\sqrt{s} - \sqrt{1-\zeta^2}}{\zeta\sqrt{s} + \sqrt{1-\zeta^2}}\right| \frac{ds}{(1+s)^{3/2}} - \frac{6MQ}{\pi^3}$$
$$\int_0^1 \arccos\sigma \frac{d\sigma}{\sigma}\int_0^\infty f(\sigma\sqrt{s+1})\ln\left|\frac{\zeta\sqrt{s} - \sqrt{1-\zeta^2}}{\zeta\sqrt{s} + \sqrt{1-\zeta^2}}\right| \frac{ds}{(1+s)^{3/2}} + O(M^2) \quad (3.12)$$

4. Determination of the size of stagnation zone. We shall use the obtained solution (3.12) of the integral equation, to find the approximate form of the stagnation zone. By (1.9) and (1.11) we have

$$\psi(u,\theta) = \frac{u^2}{4}\int_0^\infty \frac{\alpha(1+\alpha)\operatorname{sh}\theta\sqrt{\alpha}}{\operatorname{th}\sqrt{\alpha}\pi\operatorname{sh}^{1/2}\sqrt{\alpha}\pi} F(\alpha, -u)\,d\alpha \int_{a_n}^\infty (1+v)g(v)F(\alpha, -v)\,dv \qquad (4.1)$$

In order to determine the form of the boundaries of the stagnation zone, we must [1] compute

$$h(\theta) = \frac{1}{u}\frac{\partial\psi}{\partial u}\bigg|_{u=0} = \frac{1}{2}\int_0^\infty \frac{\alpha(1+\alpha)\operatorname{sh}\theta\sqrt{\alpha}}{\operatorname{th}\sqrt{\alpha}\pi\operatorname{sh}^{1/2}\sqrt{\alpha}\pi}\int_{a_n}^\infty (1+v)g(v)F(\alpha, -v)\,dv\,d\alpha \qquad (4.2)$$

Let us now make the following substitution in the last integral

$$\xi = Mm, \qquad v = (1-m)^2 / 4m \qquad (4.3)$$

and use the expression for F given by (1.14). Since the integrand appears to be an even function of $s = \sqrt{\alpha}$, we can integrate with respect to s from $-\infty$ to ∞. Using the previous asymptotic formulas for the hypergeometric functions, we obtain

$$h(\theta) = 2\int_{-\infty}^\infty s^3(1+s^2)\frac{\Gamma(2is)}{\Gamma(2+is)\Gamma(1+is)}\operatorname{cth}s\pi\,\frac{\operatorname{sh}s\theta}{\operatorname{sh}^{1/2}s\pi}(4M)^{-is} \times$$
$$\times\int_0^1 \xi^{-is}\left(\frac{1+M\xi}{1-M\xi}\right)^{3/2} G(M\xi)\left(1 + \frac{3}{2}\frac{M\xi}{1/2 - i\xi} + O(M^2)\right)\frac{d\xi}{\xi}\,ds \qquad (4.4)$$

where the integrand is analytic and decreasing in the upper semiplane (we assume that $M < 1/4$).

We can thus obtain an expression for $h(\theta)$ in the form of a series of residues at the poles of the integrand function, and they can easily be shown to be situated at the points $s = 2ik$ where $k = 1, 2, \ldots$. In consequence we obtain the following expression with the accuracy of up to the M^2-th order terms

$$h(\theta) = 64\pi^{-1}M^2 \sin 2\theta \int_0^1 Y(\xi)(1 + 0.6M\xi + O(M^2))\,\xi\, d\xi \tag{4.5}$$

Inserting here the expression obtained previously for $Y(\xi)$, we obtain

$$h(\theta) = 64\pi^{-2}QM^2 \sin 2\theta \left[\int_0^1 \zeta \arccos \zeta\,(1 + 0.6M\zeta)\, d\zeta - \right.$$

$$- \frac{3M}{2\pi}\int_0^1\int_0^\infty \ln\left|\frac{\zeta\sqrt{s} - \sqrt{1-\zeta^2}}{\zeta\sqrt{s} + \sqrt{1-\zeta^2}}\right| \frac{ds}{(1+s)^{3/2}}\,\zeta\, d\zeta -$$

$$\left.\frac{3M}{\pi}\int_0^1 \zeta\, d\zeta \int_0^1 \arccos \sigma \frac{d\sigma}{\sigma}\int_0^\infty f(\sigma\sqrt{s+1}) \ln\left|\frac{\zeta\sqrt{s} - \sqrt{1-\zeta^2}}{\zeta\sqrt{s} + \sqrt{1-\zeta^2}}\right| \frac{ds}{(1+s)^{3/2}}\right] \equiv$$

$$\equiv 64\pi^{-2}QM^2 \sin 2\theta\,(A_0 + A_1 M) \tag{4.6}$$

By [1], the points at the boundary of the stagnation zone are given by

$$x(\theta) + iy(\theta) = \lambda^{-1}\int_0^0 e^{i\varphi} h(\varphi)\, d\varphi + \text{const}$$

Substitution of (4.6) yields

$$x(0) + iy(0) = \frac{128M^2}{3\lambda\pi^2}(A_0 + A_1M + O(M^2))(-\cos^3 0 + i \sin^3 0) \tag{4.7}$$

Thus the stagnation zones vary with the parameter M with the accuracy of up to the M^2-th order terms.

Let us now determine the distance between the source and the apex of the stagnation zone, usng the Expression

$$x_0 = \frac{1}{\lambda}\int_0^\infty \frac{\partial\psi(u, 0)}{\partial 0}\frac{du}{u^2} \tag{4.8}$$

In the present case we insert here (4.1), obtaining

$$x_0 = \frac{1}{4\lambda}\int_0^\infty du \int_0^\infty \frac{\alpha^{3/2}(1+\alpha)}{\operatorname{th}\sqrt{\alpha}\pi\, \operatorname{sh}{}^1/_2\sqrt{\alpha}\pi} F(\alpha, -u)\, d\alpha \int_{a_0}^\infty (1+v)\, g(v)\, F(\alpha, -v)\, dv$$

Using now Formula

$$\int_0^\infty F(a, b, c, -z)\, dz = (c-1)(a-1)^{-1}(b-1)^{-1}$$

and transforming the resulting integrals in the manner indicated previously, we obtain

$$x_0 = \frac{2}{\lambda}\int_{-\infty}^\infty s^4 \frac{\operatorname{cth} s\pi}{\operatorname{sh}{}^1/_2 s\pi}(4M)^{-is}\frac{\Gamma(2is)}{\Gamma(2+is)\Gamma(1+is)}\int_0^1 \left(\frac{1+M\zeta}{1-M\zeta}\right)^{3/2}\zeta^{-is}G(M\zeta)\times$$

$$\times\left(1 + \frac{3}{2}\frac{M\zeta}{{}^1/_2 - is} + Q(M^2)\right)\frac{d\zeta}{\zeta}\, ds \tag{4.10}$$

This can be computed with the accuracy of up to the M^2-th order terms following the procedure used in obtaining $h(\theta)$ from (4.5), to yield

$$x_0 = \frac{4M}{\lambda}\int_0^1 Y(\zeta)\, d\zeta + \frac{4M^2}{\lambda}\left(1 - \frac{32\pi}{3}\right)\int_0^1 Y(\zeta)\,\zeta\, d\zeta \tag{4.11}$$

from which we obtain

$$x_0 = \frac{4MQ}{\pi\lambda}\int_0^1 \arccos\zeta\, d\zeta + \frac{QM^2}{\pi\lambda}\Big\{4\left(1-\frac{32\pi}{3}\right)\int_0^1 \zeta\arccos\zeta\, d\zeta -$$

$$-\frac{6}{\pi}\int_0^\infty \frac{ds}{(1+s)^{5/2}}\int_0^1 \ln\left|\frac{\zeta\sqrt{s}-\sqrt{1-\zeta^2}}{\zeta\sqrt{s}+\sqrt{1-\zeta^2}}\right| d\zeta -$$

$$-\frac{24}{\pi^2}\int_0^1 \operatorname{arc\,cos}\sigma\,\frac{d\sigma}{\sigma}\int_0^1 d\zeta\int_0^\infty f(\sigma\sqrt{s+1})\ln\left|\frac{\zeta\sqrt{s}+\sqrt{1-\zeta^2}}{\zeta\sqrt{s}-\sqrt{1-\zeta^2}}\right|\frac{ds}{(1+s)^{5/2}}\Big\} \qquad (4.12)$$

Using (4.5) and (4.12) we obtain

$$x_0 = \frac{4MQ}{\pi\lambda}(1-0.32M), \qquad x(\theta)+iy(\theta) = \frac{16QM^2}{3\pi\lambda}(1+1.00M) \qquad (4.13)$$

By (1.27) we have, for large $a_0 = a/\lambda$

$$M = {}^1/_4(\lambda/a)\,[1 + {}^3/_2(\lambda/a) + \ldots] \qquad (4.14)$$

and finally

$$x_0 = (Q/\pi a)\,(1 + 1.42\lambda/a\ldots) \qquad (4.15)$$

$$x(\theta) + iy(\theta) = {}^1/_8(Q\lambda/\pi)\,(1 + 3.2\lambda/a\ldots)(-\cos^3\theta + i\sin^3\theta)$$

5. Analysis of the structure of the approximate solution in the general case. We shall now consider a more general case. Let the problem be formulated on the hodograph plane, within a strip with one or several cuts. When $\lambda \to 0$, the resulting solution of the problem tends uniformly to a solution corresponding to the usual linear filtration law $(\lambda = 0)$ which can be obtained by conformal mapping. However, in order to define the boundary of the stagnation zone, we must calculate the limit

$$\chi(\theta) = \lim_{w\to 0}\frac{1}{w}\frac{\partial\psi}{\partial w}$$

for which the uniform convergence cases to hold. Indeed, when w is small, the solution near the line $w = 0$, $0 < \theta < \theta^*$ on which $\psi(0, \theta) = 0$, can be represented by

$$\psi(w, \theta, \lambda) = \sum_{m=1}^\infty B_m w^2 F(2-\alpha_m, 2+\alpha_m, 3, -w/\lambda)\sin\alpha_m\theta$$

$$\left(\alpha_m = \frac{\pi m}{\theta^*}\right) \qquad (5.1)$$

Since $F(2-\alpha_m, 2+\alpha_m, 3, 0) = 1$, the above limit exists for any θ and is equal to

$$2\sum_{m=1}^\infty B_m \sin\alpha_m\theta,$$

which shows that all the terms of (5.1) contribute to it. On the other hand, when $\lambda = 0$, the solution for small w can be written as

$$\psi_0(w, \theta) \equiv \psi(w, \theta, 0) = \sum_{m=1}^\infty C_m w^{\alpha_m}\sin\alpha_m\theta \qquad (5.2)$$

This implies that the limit $\chi(\theta)$ does not exist when $\pi/\theta^* < 2$, that it is equal to zero when $\pi/\theta^* > 2$ and that it has a finite value when $\theta^* = {}^1/_2\pi$. In each case the behavior at the limit is defined by the first term of (5.2).

The lack of uniformity in approximating the derivatives manifests itself only in a narrow strip adjacent to the line $w = 0$, i.e. within the boundary layer. This suggests a simple example of constructing a uniform approximation. We replace the required exact solution $\psi(w, \theta, \lambda) = \psi(w, \theta)$ with an approximate one $\psi_0(w, \theta) = \psi(w, \theta, 0)$ in the region $w \gg \lambda$ and, regarding it as a solution of (0, 1), continue it into the domain of small values of w. This method can be used whenever the ratio a/λ is sufficiently large and yields only the principal term of the asymptotic equation. Generally speaking, only this term is independent of the position of the auxiliary boundary on which the exact solution is replaced by an approximate one (of course this boundary must lie within the region $w \gg \lambda$).

Example. Let us seek a solution of (0, 1) on the hodograph plane within the semistrip $0 < w < \infty$, $0 < \theta < \theta_0$ with one (or more) segment (segments) $0 < w < a$, $\theta = \theta_1$ eliminated. Let us put

$$\psi_0(w, \theta_1) = g(u) \qquad (w = u\lambda \geqslant \alpha = a_0\lambda)$$

where $\psi_0(w, 0)$ denotes the solution corresponding to $\lambda = 0$, and let us assume approximately

$$\psi(w, \theta_1) \equiv \psi(w, \theta_1, \lambda) = g(u) \qquad (u > a_0) \tag{5.3}$$

This will divide the semistrip $0 < \theta < \theta_0$ into two parts, and the boundary values of the function ψ will be given in each of them. The resulting simplified problem can be solved using the integral transformation (1, 5). In particular, for the solution $\psi(w, \theta)$ in the semistrip $0 < \theta < \theta_1$ we have

$$\psi(w, 0) = \psi(0, \theta) = 0, \quad \psi(w, \theta_1) = 0 \ (w < a) \quad \psi(w, \theta_1) = g(u) \ (u > a_0)$$

and similarly to (4, 1),

$$\psi(u, \theta) = \frac{u^2}{4} \int_0^\infty \frac{\alpha(1+\alpha)}{\operatorname{th}\sqrt{\alpha}\pi} \frac{\operatorname{sh}\sqrt{\alpha}\theta}{\operatorname{sh}\sqrt{\alpha}\theta_1} F(\alpha, -u)\, d\alpha \int_{a_0}^\infty (1+v)\, g(v)\, F(\alpha, -v)\, dv \tag{5.4}$$

To find the boundary of the stagnation zone corresponding to the segment $w = 0$, $0 < \theta < \theta_1$, we can simplify Expression (5, 4) even further, repeating the procedure given in Section 4. The only difference lies in the fact, that the pole of the integrand making the fundamental contribution to the solution, is situated at the point $s = i\pi / \theta_1$. As the result, we have

$$h(\theta) = -\frac{4\pi^2}{\theta_1^2}\left(1 + \frac{\pi}{\theta_1}\right)(4M)^{\pi/\theta_1} \frac{\Gamma(-2\pi/\theta_1)}{[\Gamma(-\pi/\theta_1)]^2} \frac{\sin \pi\theta/\theta_1}{\operatorname{tg}\pi^2/\theta_1} \int_0^1 \xi^{\pi/\theta_1 - 1} G(M\xi)\, d\xi$$

By (1, 27) we have $M \sim 1/4\ a_0$ as $a_0 \to \infty$, hence (5, 4) finally yields

$$h(\theta) = -\frac{4\pi^2}{\theta_1}\left(1 + \frac{\pi}{\theta_1}\right) \frac{\Gamma(-2\pi/\theta_1)\, a_0^{-\pi/\theta_1}}{[\Gamma(-\pi/\theta_1)]^2} \frac{\sin \pi\theta/\theta_1}{\operatorname{tg}\pi^2/\theta_1} \int_0^1 \xi^{\pi/\theta_1 - 1} g\left(\frac{a}{\xi}\right) d\xi \tag{5.5}$$

from which we obtain the following expression for the coordinates of the stagnation zone:

$$x(\theta) + iy(\theta) = \frac{4\pi^2 a_0^{-\pi/\theta_1}}{\theta_1(\pi - \theta_1)\lambda} \frac{\Gamma(-2\pi/\theta_1)}{[\Gamma(-\pi/\theta_1)]^2} \int_0^1 \xi^{\pi/\theta_1 - 1} g\left(\frac{a}{\xi}\right) d\xi \times \tag{5.6}$$

$$\times \operatorname{ctg}\frac{\pi^2}{\theta_1}\left[\left(\frac{\pi}{\theta_1}\cos\frac{\pi\theta}{\theta_1}\cos\theta + \sin\theta\sin\frac{\pi\theta}{\theta_1}\right) + i\left(\cos\theta\sin\frac{\pi\theta}{\theta_1} - \frac{\pi}{\theta_1}\sin\theta\cos\frac{\pi\theta}{\theta_1}\right)\right]$$

The distance between the tip of the stagnation zone and the source is equal, with the accuracy of up to the terms tending to zero with decreasing λ, to the distance between the critical point of the flow ($w = 0$) and the source, provided that the linear filtration law is obeyed. This result which is obvious, can be obtained from the formulas analogous to (4, 9) and (4, 10).

2°. We shall in addition consider a flow generated by a source-sink combination of equal strength. In this case an outer stagnation zone is formed in the physical plane (Fig. 2 of [1]) and the corresponding problem in the hodograph plane is that of obtaining a solution of Eq. (0.1) on the semistrip $0 < w < \infty$, $0 < \theta < 2\pi$ with a cut $\theta = \pi$, $a < w < \infty$. We have $\psi = 0$ on the outer boundary and $\psi = Q$ along the cut. Let us produce an auxiliary boundary $\lambda \ll w = a' \leqslant a$, $0 < \theta < 2\pi$ and denote

$$\psi(a', \theta) = f(\theta) \tag{5.7}$$

Solution of (0.1) in the rectangle $0 < \theta < 2\pi$ $0 < w < a'$ can be obtained by the Fourier's method, and has the form

$$\psi(w, \theta) = \sum_{m=1}^{\infty} f_m \left(\frac{w}{a'}\right)^2 \frac{F(2 - {}^1/_2 m,\ 2 + {}^1/_2 m,\ 3,\ -w/\lambda)}{F(2 - {}^1/_2 m,\ 2 + {}^1/_2 m,\ 3,\ -a'/\lambda)} \sin\frac{m\theta}{2} \tag{5.8}$$

where f_m are the coefficients of expansion of $f(\theta)$ into a sine series on the segment $(0, 2\pi)$.

As the result, we have the following expression for the boundaries of the stagnation zone:

$$x(\theta) + yi(\theta) = x(0) + \frac{4\lambda}{a'^2} \sum_{m=1}^{\infty} \frac{f_m}{F(2 - {}^1/_2 m,\ 2 + {}^1/_2 m,\ 3,\ -a'/\lambda)} \times$$

$$\times \left[\frac{m}{m^2 - 4} - \frac{m\cos\theta\cos{}^1/_2 m\theta + 2\sin\theta\sin{}^1/_2 m\theta}{m^2 - 4} - \right.$$

$$\left. - i\,\frac{m\sin\theta\cos{}^1/_2 m\theta - 2\cos\theta\sin{}^1/_2 m\theta}{m^2 - 4}\right] \tag{5.9}$$

This can be simplified for small λ by considering the asymptotic behavior of the hypergeometric function appearing in the denominator, as $a'/\lambda \to \infty$. When $m < 4$, then the asymptotic equation follows from the formula for the analytic continuation ([6], Formula 2.1 (18))

$$F\left(2 - \frac{1}{2}m;\ 2 + \frac{1}{2}m,\ 3,\ -\frac{a'}{\lambda}\right) \sim \left(\frac{a'}{\lambda}\right)^{-2 + {}^1/_2 m} \frac{2\Gamma(m)}{\Gamma(1 + {}^1/_2 m)\,\Gamma(2 + {}^1/_2 m)} \tag{5.10}$$

When m are large and even, F becomes a polynomial in (a'/λ) of degree ${}^1/_2 m - 2$; for odd m, the asymptotic formula (5.10) holds. Thus, (5.9) yields the expansion in terms of functions of λ of order increasing with m and the principal term of the expansion has the form

$$x(\theta) + iy(\theta) = x(0) + \frac{64 f_1}{9(\lambda a')^{1/2}\pi} \times$$

$$\times \left[\cos\theta\cos\frac{\theta}{2} + 2\sin\theta\sin\frac{\theta}{2} - 1 + i\left(\sin\theta\cos\frac{\theta}{2} - 2\cos\theta\,\sin\frac{\theta}{2}\right)\right] \tag{5.11}$$

When computing this term, we can simplify the expression for f_1, by taking it for the case $\lambda = 0$. We then have

$$\psi_0(w, \theta) = \frac{Q}{2} + \operatorname{Im}\frac{Q}{\pi i}\arcsin\left(1 + \frac{2w}{a}e^{i\theta}\right) \qquad (0 < \theta < \pi)$$

$$f_0(\theta) = \psi_0(a, \theta) = \frac{Q}{2} + \operatorname{Im}\frac{Q}{\pi i}\arcsin\left(1 + \frac{2a'}{a}e^{i\theta}\right) \tag{5.12}$$

from which we obtain

$$f_1 = \frac{Q}{\pi}\int_0^{2\pi} f_0(\theta)\sin\frac{\theta}{2}\,d\theta = Q\sqrt{\frac{a'}{a}}$$

Inserting this into (5.11) we see that the result is, as expected, independent of the choice of the boundary a' and is equal to

$$x(\theta)+iy(\theta)=x(0)+\frac{64}{9\pi}\frac{Q}{(\lambda a)^{1/2}}\times$$
$$\times\left[\cos\theta\cos\frac{\theta}{2}+2\sin\theta\sin\frac{\theta}{2}-1+i\left(\sin\theta\cos\frac{\theta}{2}-2\cos\theta\sin\frac{\theta}{2}\right)\right] \quad (5.13)$$

Before we return to the physical plane, we must establish the connection between the auxiliary parameter a and the distance $2L$ between the source and the sink. We have (see e.g. [1])

$$L=\int_a^\infty \frac{\partial\psi}{\partial\theta}\frac{dw}{w^2} \quad (5.14)$$

Obviously, the principal term of L can be obtained under the assumption that $\lambda=0$, i.e.

$$L=\int_a^\infty \frac{\partial\psi_0(w,\pi)}{\partial\theta}\frac{dw}{w^2}=\frac{Q}{\pi}\int_a^\infty\frac{dw}{w^{3/2}\sqrt{a+w}}=\frac{2(\sqrt{2}-1)Q}{\pi a} \quad (5.15)$$

and (5.13) can then be written as

$$x(\theta)+iy(\theta)=\frac{32}{9}\left(\frac{2(1+\sqrt{2})QL}{\pi\lambda}\right)^{1/2}\times \quad (5.16)$$
$$\times\left[\cos\theta\sin\frac{\theta}{2}+2\sin\theta\sin\frac{\theta}{2}+i\left(\sin\theta\cos\frac{\theta}{2}-2\cos\theta\sin\frac{\theta}{2}\right)\right]$$

where $x(0)$ is computed under the symmetry condition $x(0)=-x(2\pi)$.

3°. We can find how small λ must be in order to make the principal term sufficient by calculating this term in a problem possessing an exact solution.

Consequently, we shall consider the flow due to an infinite row of identical sources of strength $4Q$ (see [2]). On the hodograph plane, this problem reduces to that of obtaining a solution on the semistrip $0<\theta<{}^1/_2\pi$, $0<w<\infty$ under the following conditions:

$$\psi(w,\theta)=\psi(0,\theta)=0,\quad \psi(w,{}^1/_2\pi)=0,\ (w<a),\quad \psi(w,{}^1/_2\pi)=Q\ (w>a)$$

Here the value of g is given and the principal term of the series solution is obtained at once by inserting $g=Q$ into (5.6). This yields

$$x(\theta)+iy(\theta)=\lim_{\theta_1\to{}^1/_2\pi}\left[\frac{8Q\pi}{\pi-\theta_1}\frac{\Gamma(-2\pi/\theta_1)}{[\Gamma(-\pi/\theta_1)]^2}\operatorname{ctg}\frac{\pi^2}{\theta_1}\right]\times$$
$$\times\left[(2\cos 2\theta\cos\theta+\sin\theta\sin 2\theta)+i(\cos\theta\cos 2\theta-2\sin\theta\cos 2\theta)\right]=$$
$$=\frac{4Q\lambda}{3\pi a^2}(-\cos^3\theta+i\sin^3\theta)$$

from which we obtain, using the relation $a=Q/L$, where L is the half distance between two neighboring sources,

$$\frac{x(\theta)+iy(\theta)}{L}=\frac{4\lambda L}{\pi Q}(-\cos^3\theta+i\sin^3\theta) \quad (5.17)$$

Fig. 1 shows the approximate (solid lines) and exact (broken lines) solutions. The accompanying numbers denote the corresponding values of the parameter $a_0=Q/\lambda L$.

4°. When λ is sufficiently small, the same method can be applied to more complex flows for which one to one mapping to and from the hodograph plane is not possible. Here stagnation zones may appear either near the critical points of the unperturbed flow ($\lambda=0$) or near the point at infinity, i.e. near those points of the unperturbed flow, at which the velocity becomes zero and λ is real, however small. A circle can be drawn around each critical point, on which the velocity w will have a constant value $a\gg\lambda$. and the distribution of the stream function and velocity on these circles can be assumed to be the same as that at $\lambda=0$. The flow within the circles will possess a localized

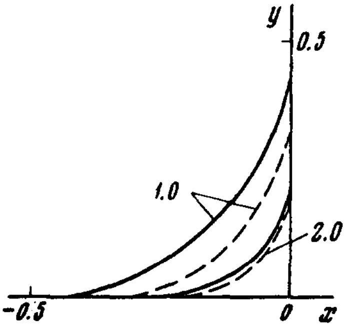

Fig. 1

symmetry and will consist of a number of equal sectors each of which can be mapped on a rectangle in the hodograph plane. Thus it is easy to obtain an approximate flow pattern and stagnation zones in the isolated regions. The outer stagnation zones surrounding the point at infinity are, of course, the most interesting ones, since the stagnation zones near the inner critical points of the flow are small, when λ is small.

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ANALYSIS OF SECONDARY STEADY FLOW BETWEEN ROTATING CYLINDERS

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Appearance of a secondary steady flow which is a Taylor vortex, caused by the loss of stability of the Couette flow between the rotating (in the same direction) cylinders is investigated using the Liapunov-Schmidt method. It is shown that the secondary solution can be obtained in the form of a series in powers of the parameter $\varepsilon = (N_{\mathrm{Re}} - N_{\mathrm{Re}*})^{1/2}$ where N_{Re} is the Reynolds' number and $N_{\mathrm{Re}*}$ denotes its critical value. First two terms of the series are analised for two separate cases and it is established that the Taylor vortex is defined uniquely with the accuracy of up to the displacement in the axial direction. Perturbation theory is used to show that at small ε the Taylor flow is stable with